A GENERAL THEOREM RELATING TO TRANSVERSALS, AND ITS CONSEQUENCES.

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[Continued from Vol. X, p. 192.]

It is evident that some of the properties expressed in the preceding theorems are projective and therefore true for all conics. And it can easily be shown that a much larger number of these properties remain true when subjected to the process of "orthogonal" projection.

First, let us suppose all of the preceding figures to be projected orthogonally upon any plane. The "given circle" will project into a "given ellipse" and all other circles will project into "similar ellipses with axes parallel to the axes of the given ellipse," and straight lines into straight lines.

Since the orthogonal projection of a finite line upon any plane is equal to the line multiplied by the cosine of the angle which the line makes with the plane, it follows that if each of the segments, made by the sides of the quadrilateral and this system of ellipses upon any transversal through a D-point, be multiplied by the cosecant of the angle which this line makes with the projected transversal, the results may be substituted in any of the foregoing equations. If equations (8), (10), (12), (14), (18), (20), (23), (24), (24a), (27), (28), (35), (36) be treated in this way it will be found that in any equation this constant factor appears to the same degree in every term, and hence the new equation is of the same form as the original.

∴ These relations are also true in the projected figure. Therefore, in order to state a particular case of the corresponding theorems for a quadrilateral inscribed in an ellipse, it is only necessary to substitute "given ellipse" for "given circle" and in all other places for "circles" to write "similar ellipses with axes parallel to the axes of the given ellipse;" and in (5), of Theor. I, to omit the clause concerning the radius of the foci locus, and also to strike out the last statement of (2) in Theor. III. All the other conclusions are still valid.

The results obtained by conical projection are not so simple. Here it is necessary to observe that all circles pass through the same two points at infinity. Therefore these circles must project into a system of conics through two fixed points, and each conic otherwise fulfils the same conditions as the circle of which it is the projection. Thus, we derive the following for conics in general:—

If a quadrilateral is inscribed in a conic and on each of its sides a conic is described passing through a D-point and all intersecting the given conic in the same two points, and any right line is drawn through this D-point:

We have for conclusions (4), (5) (omitting the last clause), (7), (8) of Theor. I, and (2), (3) of Theor. II, by substituting for "circles," "conics intersecting the given conic in the same two points."

For (1) of Theor. II, we must write: "In a crossed quadrilateral three of these conics meet their respective opposite sides in six points which lie on another conic passing through the same two common points."

The projection of Theor. III and Fig. 8 requires a separate and fuller statement. The circles project into conics intersecting the given conic in the same two points as before.

Since the center of a circle is the pole of the line at infinity, the center of the given circle projects into the pole of the common chord with respect to the given conic, and similarly the center of each of the other circles projects into the pole of the same line with respect to the conic into which that circle projects.

We thus obtain the following theorem :—

If a triangle is self-polar with respect to a given conic, and if conics are described on each of its sides intersecting the given conic in two common points, the pole of the common chord for each of these conics lying on the side of the self-polar triangle upon which the conic is described ; and if three other conics are drawn intersecting the given conic in the same two points, and passing through the pole of this common chord with respect to the given conic, and each passing through one vertex of the self-polar triangle and having the pole of the common chord with respect to itself lying on the right line joining this vertex to the pole of the common chord with respect to the given conic, then :

1. Every straight line drawn through the pole of the common chord with respect to the given conic is cut by these six conics and the three sides of the self-polar triangle in an involution of which this fixed pole is the center.

2. The given conic is the locus of the foci of this involution as the transversal is supposed to revolve about this fixed pole.

3. Every chord intercepted on this variable transversal by the given conic is divided harmonically (1) by each of the conics upon the sides of the self-polar triangle (provided they cut the transversal), and (2) by each side of the self-polar triangle and the conic passing through the opposite vertex and the pole of the common chord with respect to the given conic.

4. Four of these conics pass through each of the points of intersection of the sides of the self-polar triangle with the lines joining its vertices to the

pole of their common chord with respect to the given conic, and by joining these points each becomes the vertex of a harmonic pencil.

Some of the results which were proved in connection with the discussion of Fig. 8 are susceptible of the following interpretation, which from a careful examination of the figure may easily be seen to be true.

THEOREM IV. A line drawn through the vertex of any triangle is cut in involution by the circle described upon the opposite side as a diameter, the circumcircle, and the lines joining the feet of the altitudes from the other vertices, a circle whose diameter is the line from this vertex to the intersection of the altitudes and the opposite side, and circles described upon adjacent sides as diameters, and the altitudes upon those sides. The vertex through which this variable transversal is drawn is the center of the involution (and the locus of the foci is a circle with the same center, and a radius equal to a mean proportional between an adjacent side and the distance to the foot of the altitude upon that side).

A line drawn through the intersection of the altitudes is also cut in involution by the three circles upon the sides as diameters, the three circles having for diameters the lines joining this point to the vertices and the sides of the triangle.

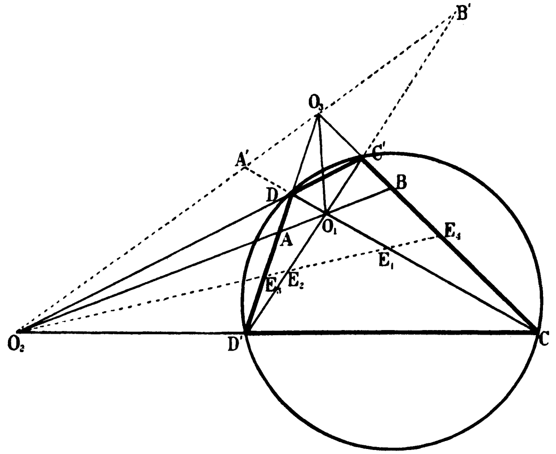

Fig. 9.

This theorem is also orthogonally and conically projective, excepting the part in the parenthesis.

Let us now resume equation (20) and apply it to the complete quadrilateral of Fig. 9.

Draw any line through O_2 cutting the sides in E_3, E_2, E_1, E_4, and let $O_2E_2 \equiv p$, $O_2E_1 \equiv q$, $O_2E_4 \equiv r$, $O_2E_3 \equiv s$.

Now suppose this line to revolve about O_2 until it passes through O_1. Then E_1, E_2 coincide, $q = p$, and (20) reduces to

$$p\,(r + s) = 2rs\,, \tag{41}$$

whence

$$\frac{r - p}{p - s} = \frac{r}{s}, \tag{42}$$

$$\therefore \qquad \frac{O_1B}{O_1A} = -\frac{O_2B}{O_2A}, \quad \frac{O_2A\,.\,O_1B}{O_1A\,.\,O_2B} = -1\,. \tag{43}$$

By supposing the line to revolve until it passes through O_3 we get in the same way

$$\frac{O_3B'}{O_3A'} = -\frac{O_2B'}{O_2A'}, \quad \text{or} \quad \frac{O_2A'\,.\,O_3B'}{O_3A'\,.\,O_2B'} = -1\,. \tag{44}$$

That is, according to our present conception of the quadrilateral, the lines joining the intersections of opposite sides are divided harmonically by the diagonals, and hence at each D-point the two diagonals together with the two sides of the triangle of the D-points form a harmonic pencil.

This furnishes a simple proof of the familiar theorem that in a complete quadrilateral (inscribed in a circle) each diagonal is divided harmonically by the other two. But we have also shown that the relation expressed in (20) is

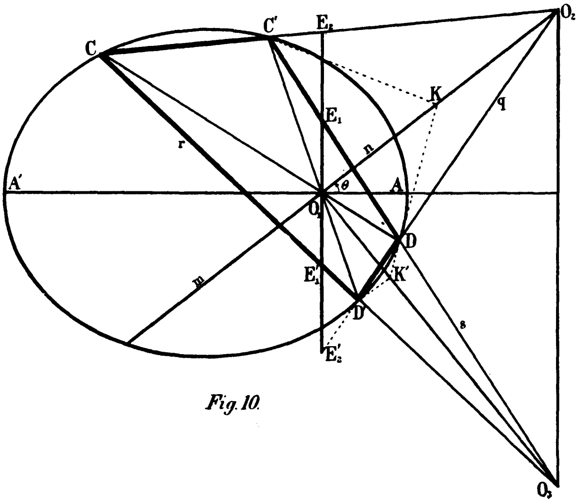

Fig. 10.

projective, orthogonally, and consequently it is true for any quadrilateral inscribed in an ellipse, that is, for any convex quadrilateral. Therefore, in any complete convex quadrilateral each diagonal is divided harmonically by the other two.

Consider next this equation in its application to a quadrilateral inscribed in an ellipse with the interior D-point at one focus.

Let m, n represent the segments of the variable focal chord, and p, q, r, s, the segments intercepted by the sides upon which these letters are written in Fig. 10. Then, for O_2, (20) takes the form

$$\frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s}. \tag{45}$$

By means of the polar equation of the ellipse it is easily shown that

$$\frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s} = \frac{a(1-e^2)}{2e\cos\theta}, \tag{46}$$

where θ is the angle the segment n makes with the major axis. From this equation, it is seen that these variable ratios are equal to the distance on the variable transversal between the focus and a line perpendicular to the axis at a point midway between the focus and the directrix; for multiplying (46) by two and placing the results equal to ρ gives

$$\rho = \frac{2mn}{m-n} = \frac{2pq}{p-q} = \frac{2rs}{r-s} = \frac{a(1-e^2)}{e\cos\theta}, \tag{47}$$

which is the polar equation of the directrix, the focus being taken as the pole.

When $\theta = 90°$

$$\rho = \frac{2mn}{m-n} = \frac{2pq}{p-q} = \frac{2rs}{r-s} = \infty. \tag{48}$$

$$\therefore \qquad m = n,\ p = q,\ r = s, \quad \text{i. e.,} \quad O_1E_1 = O_1E_1',\ O_1E_2 = O_1E_2'. \tag{49}$$

Therefore right lines drawn through the opposite ends of two focal chords of an ellipse intersect the latus rectum at equal distances from the focus.

When q is positive and equal to p and when s is positive and equal to r, we have, respectively,

$$\rho_1 = p = \frac{a(1-e^2)}{e\cos\theta_1}, \quad \text{and} \quad \rho_2 = r = \frac{a(1-e^2)}{e\cos\theta_2}. \tag{50}$$

Hence the opposite sides intersect on the directrix, and since the triangle of the D-points has been shown to be self-polar, it follows that the directrix is the polar of the focus. Of course this result is well known, but it serves as an important check for the validity of this chain of demonstrations.

Also, tangents at the extremities of these focal chords intersect on the sides of the self-polar triangle; and the line joining the focus to the intersec-

tion of two tangents bisects the angle between the focal radii of the points of tangency.*

Therefore the self-polar triangle is right-angled at the focus.

Some interesting consequences can now be deduced by inverting Fig. 10 using the focus as the center of inversion.

We thus obtain the *limaçon* of Fig. 11 with circles through the focus and the adjacent extremities of two focal chords.

Putting m and n for the segments intercepted by the limaçon on a variable focal chord, and p, q, r, s for the segments intercepted by the circles whose centers are marked by these letters, respectively, and remembering that these

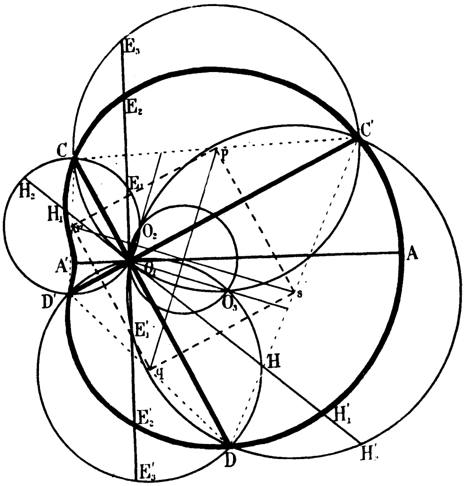

Fig. 11.

are the reciprocals of the corresponding lines of Fig. 10, we get, by substituting in (47) and reducing and also taking the reciprocal of ρ

$$\rho = \tfrac{1}{2}(n - m) = \tfrac{1}{2}(q - p) = \tfrac{1}{2}(s - r) = \frac{e \cos \theta}{a(1 - e^2)}. \tag{51}$$

This is the equation of the circle $O_2 O_1 O_3$, which, as it is the inverse of the directrix, may be called the "Directrix Circle." It will be observed that this result is exactly the same in form as (23).

When $\theta = 90°$, $2\rho = (n - m) = (q - p) = (s - r) = 0$,

$$\therefore \qquad n = m,\ q = p,\ s = r, \quad \text{i. e.,} \quad O_1E_1 = O_1E_1',\ O_1E_3 = O_1E_3'. \tag{52}$$

* Newcomb's Anal. Geom., p. 166, Ex. 54.

Therefore when circles are drawn through the focus and the ends of two focal chords of a limaçon, opposite circles intersect the parameter at equal distances from the focus.

When the variable focal chord is tangent to one circle, say p, its intercept reduces to zero, and we have

$$n - m = q = s - r, \text{ and } \therefore\ n - q = m,\ s - q = r,\ r - m = s - n, \tag{53}$$

or, as is shown on the figure

$$O_1H_1 = HH_1',\ O_1H_2 = HH_2',\ H_1H_2 = H_1'H_2'.$$

Hence, when the variable focal chord is tangent to one circle, it is intersected by the opposite circle, and also by the other two circles, at equal distances from its intersection with the limaçon.

As p, r in turn become positive and equal to q, s respectively we get from (51)

$$\rho_1 = q = \frac{e \cos \theta_1}{a(1 - e^2)}, \quad \rho_2 = s = \frac{e \cos \theta_2}{a(1 - e^2)}, \tag{54}$$

which shows that opposite circles intersect on the directrix circle. This fact also follows directly, for O_2 and O_3 are the inverse points of the two exterior D-points of Fig. 10, and hence furnishes another check for the truth of our results. Consequently the lines of centers of opposite circles, pq and rs, pass through the center of the directrix circle and are perpendicular to O_1O_3, O_1O_2, respectively; but these last two lines are the sides of the self-polar triangle of Fig. 10, which, as was stated before, are at right angles. Hence pq is perpendicular rs. Also the lines of centers of adjacent circles are perpendicular to the fixed focal chords, since the focal radii are the common chords of these pairs of circles.

Therefore the centers of these four circles are the vertices of a rhombus whose center is the center of the directrix circle.

As a special case of this the following may be noted :

If a quadrilateral is formed by joining the ends of two perpendicular focal chords of a limaçon, the lines joining consecutively the middle points of its sides enclose a square whose center is the center of the directrix circle and whose side is equal to half the parameter. Fig. 11 exemplifies this special case.

In the discussion of Fig. 10, we might have included the other similar ellipses drawn through the focus and the vertices of the quadrilateral; these would invert into cubics (since the center of inversion is on the conics) through the ends of the focal chords of the limaçon. We should thus have, as before,

the variable transversal in Fig. 11 cut in involution by the limaçon, the four circles, and the four cubics. The special results as to concurrence and collinearity would be the same as in former figures. However, one consequence deserves special attention.

In these involutions conjugate points lie respectively on a circle and the opposite cubic; also the conjugate of the center is at infinity.

Therefore, the tangents to the circles at the focus of the limaçon are the asymptotes of the cubics.

Since the parabola is but the special case of the ellipse, with one focus removed to infinity, it follows that all these results will remain true if we substitute the word "parabola" for "ellipse." For precisely the same reason we may write "cardioid" in the place of "limaçon."

Now suppose that a given circle intersects a given ellipse (or parabola) in four real points. Draw the quadrilateral determined by the four points of intersection. Also draw the three systems of circles as in Fig. 7, and the three systems of similar ellipses (or parabolas) as stated in the theorem derived by orthogonal projection. Let the small letters denote the intercepts of the quadrilateral and the circles as in Fig. 6, and let the corresponding capitals represent the intercepts made by the corresponding conics, that is, conics through the same three points as the circles, respectively.

Then we have for the interior D-point

$$\frac{MN}{M-N} = \frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s}, \tag{55}$$

$$mn = p'q = pq' = r's = rs', \tag{56}$$

$$MN = Pq = pQ = Rs = rS, \tag{57}$$

$$\therefore \quad \frac{mn}{MN} = \frac{p'}{P} = \frac{q'}{Q} = \frac{r'}{R} = \frac{s'}{S} = \text{a variable}. \tag{58}$$

The following conclusions are now apparent:

1. Intercepts made by the given circle, the given conic, opposite sides of the quadrilateral, opposite circles, and opposite conics are simultaneously equal.

2. The intercepts of a circle and a conic through the same three points are proportional to those of any other circle and conic through three common points, and also proportional to the products of the segments made by the given circle and the given conic.

3. For parabolas M (or N), P, Q, R, S become infinite simultaneously.

4. In (58) the terms of any fraction must reduce to zero simultaneously.

Therefore, a circle and a conic through the same three points have a common tangent at the D-point which is parallel to the opposite side of the quadrilateral.

Now conceive the seven circles of Fig. 8 to be drawn on the same diagram, and the same number of similar ellipses (or parabolas) with axes parallel to the axes of the given ellipse (or parabola) to be drawn through the same three points, respectively, as the circles. We thus add four members to each of the equations (56), (57), (58), and extend these relations to the intercepts of nine circles and nine ellipses (or parabolas). Evidently the same equations are true for either of the exterior D-points. The diagram would now contain twenty circles and twenty similar ellipses (or parabolas) with parallel axes, all of which are connected by these relations, as the transversal is drawn through the different D-points. No attempt has been made to draw this figure.

These properties are also orthogonally projective, and by this means we obtain a figure which consists of a group of twenty similar ellipses with parallel axes, and another group of twenty similar ellipses (or parabolas) with parallel axes, but whose axes are not necessarily parallel to those of the first group.

Therefore equations (55) to (58) are true when the quadrilateral is formed by joining the four real points of intersection (1) of two ellipses (2) of an ellipse and a parabola (3) of two parabolas, and the interpretation is the same as that already given for the circle and the ellipse.

When the given conics are both parabolas their axes cannot be parallel. In this case the numerators of (58) become simultaneously infinite as well as the denominators.

As a special case we may suppose the given circle to intersect the given ellipse (or parabola) so that the interior D-point shall coincide with the focus. Then using the focus as center of inversion we get a quadrilateral inscribed in both a circle and a limaçon (or cardioid), a system of four circles through the focus, and a system of four cubics through the vertices of the quadrilateral, inverting only the two systems of curves through the focus of the conic.

The inverse of (55) takes the form

$$M - N = m - n = p' - q' = r' - s' , \tag{59}$$

which should be compared with (23) and (51).

Equations (56), (57) invert respectively into

$$mn = pq' = p'q = rs' = r's , \tag{60}$$

$$MN = Pq' = p'Q = Rs' = r'S , \tag{61}$$

$$\therefore \qquad \frac{mn}{MN} = \frac{p}{P} = \frac{q}{Q} = \frac{r}{R} = \frac{s}{S} . \tag{62}$$

The small letters have their usual significance, and M, N denote the segments of the variable transversal made by the quartic, and P, Q, R, S, denote the intercepts of the four cubics.

The meaning of this result is evident.

These results will doubtless suggest to the reader the familiar theorem that "A system of conics passing through four fixed points meets any transversal in a system of points in involution."

The involutions here considered are not special cases of that stated in the theorem though they are closely allied to it, as the centers of the former are coincident conjugate points, and therefore foci, of the latter, when the two transversals coincide.

The subject is not yet exhausted, but the discussion has been carried far enough to illustrate the resources and advantages of the modern geometrical methods.